

# Nucleation Rate of Hadron Bubbles in Baryon-Free Quark-Gluon Plasma

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We evaluate the factor  $\kappa$  appearing in Langer's expression for the nucleation rate extended to the case of hadron bubbles forming in zero baryon number cooled quark-gluon plasma. We consider both the absence and presence of viscosity and show that viscous effects introduce only small changes in the value of  $\kappa$ .

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## I. INTRODUCTION

There has been a great deal of recent activity related to the study of the quark-hadron phase transition. For the transition from quark matter to hadron matter there is interest in the rate of formation of critical size hadron bubbles, specifically bubbles of pion gas, in a supercooled vapor [1,2]. The nucleation rate is an essential ingredient of the complete treatment of a phase transition, as critical size bubbles must first be formed in the quark vapor before the vapor can be converted into hadron material. Hadron bubbles initially form due to fluctuations of the energy density in the vapor. Those with radii,  $R$ , smaller than a critical size  $R_o$  collapse, while those of the critical size begin to grow exponentially. Langer and Turski [3] have shown that the nucleation rate,  $I$ , can be written as follows:

$$I = \frac{\kappa}{2\pi} \Omega_o \exp(-\Delta F/T). \quad (1)$$

Here  $\kappa$  is related to the growth rate of the radius,  $R$ , of the bubble near the critical radius,  $R_o$ , by  $dR/dt \approx \kappa(R - R_o)$ . The factor  $\Omega_o$  is a statistical prefactor, and  $\Delta F$  the difference in free energy of systems with, and without, a critical size drop present. This paper is concerned primarily with discussing the rate  $\kappa$ .

Csernai and Kapusta [1] have applied the Langer-Turski nucleation rate, (1), to the formation of hadron bubbles in a quark vapor. They consider both the hadron and quark materials as substances with zero baryon number and treat them using relativistic formalism. One principal result of [1] is the suggestion that for baryon-free matter  $\kappa$  is proportional to the viscosity coefficients of the quark vapor, and thus when these coefficients vanish a hadron bubble does not form.

We reexamine the relativistic evaluation of  $\kappa$  in this paper, and show that viscosity is not necessary for the growth of hadron bubbles, and that its effects are of higher order than the energy flow terms in the growth process. The difference between our results and those of Ref. [1] seems to arise from the conceptual distinction

between heat conduction and energy flow. The former is, indeed, not defined in baryon free systems i.e. systems with a chemical potential of zero as pointed out in [1] and other sources. Energy flow, however, is defined. A difference between our results and Ref. [1] also arises from the technical difference in the treatment of the pressure gradients.

The paper is organized as follows. In Section 2., we review the treatment of energy flow, viscosity and thermal conduction in the baryon, and baryon-free, cases of relativistic hydrodynamics. We consider the distinction between heat conduction and energy flow and stress that the latter can occur without viscosity in the baryon-free case. In Section 3., we calculate the rate  $\kappa$  in the absence of viscosity and baryon number by solving relativistic energy and momentum transport equations. The energy and momentum relativistic equations are mathematically of the same form as the mass conservation and momentum nonrelativistic transport equations for a nonviscous vapor with perfect heat conduction. We compare the relativistic result with the solution for  $\kappa$  in such a nonrelativistic case as obtained in [3], and show that the forms are similar. The method of solution of the transport equations used in this paper differs from that used in Ref. [3]. We compare the two methods, both of which can be used in either the relativistic and nonrelativistic cases. In Section 4, we reexamine the calculation of the rate  $\kappa$  in [1], focusing on the relationship between the free energy and the gradient of pressure, which is used in the momentum transport equation. In particular, we suggest that a source for the difference between the conclusion of this work and those of Ref. [1], is linked to a difference in treatments of the pressure gradients. Finally, in Section 5, we examine the relativistic transport equations in the presence of viscosity and suggest that viscous effects are of high order in the case of small viscosity coefficients.

## II. ENERGY VERSUS HEAT FLOW IN BARYON AND BARYON-FREE RELATIVISTIC HYDRODYNAMICS

We first review the transport relationships for a system with nonzero baryon number. This is a system for which the baryon chemical potential is also nonzero. Under this condition there is a net number of either particles or antiparticles present. Thus one can define a number density,  $n$ , and obtain the conservation of net particle (or antiparticle) number in the absence of dissipative effects:

$$\frac{\partial(nu^i)}{\partial x^i} = 0. \quad (2)$$

Here  $u^i$  represents the 4-velocity (divided by the velocity of light) of the net particles. The energy density and flow are given by the stress tensor (also in the absence of dissipative effects):

$$T^{ij} = pg^{ij} + (e + p)u^i u^j \quad (3)$$

which provides energy and momentum conservation through the relationship:

$$\frac{\partial T^{ij}}{\partial x^i} = 0. \quad (4)$$

Here  $g^{ik}$  is the metric,  $e$ , the energy density,  $p$ , pressure, and  $u_0 = 1/\sqrt{1 - v^2/c^2} = \gamma$ . The combination  $e + p$  is the enthalpy density,  $w$ . Both the energy flow and particle flow are related to  $u^i$ . A point we wish to emphasize is that the term  $wu^0 u^i$  represents the energy flux in the absence of viscosity. The enthalpy density depends on the material (particles and antiparticles) in the respective volume.

For the case of nonzero dissipation, an additional term  $\tau^{i0}$  containing viscosity coefficients (see [5]) is added to  $T^{i0}$  in (3). The new tensor still satisfies (4). However, the particle conservation relationship (2) is replaced by:

$$\frac{\partial(nu^i + \nu^i)}{\partial x^i} = 0 \quad (5)$$

where  $\nu^i$  is proportional to the coefficient of thermal conduction [5]. In the relativistic dissipative case it is convenient to associate  $u^i$  with the velocity of energy flow, and to note that an energy flux necessarily involves a mass flux [5].

Next we examine the case of zero baryon number, i.e. the case of zero chemical potential and  $n = 0$ . In this case there is no net particle number and the thermal conductivity is not defined, because there is no net baryon density with respect to which energy can be conducted. The energy flow, however, need not be zero. The energy flow is explicitly contained in (4). In the low velocity limit, i.e. to first order in  $v$  and in the absence of viscosity, equation (4) gives:

$$\partial_t e = -\nabla \cdot (w\vec{v}) \quad (6)$$

and for  $i \neq 0$

$$\partial_t w\vec{v} = -\nabla p. \quad (7)$$

Here, the velocity of light has been set equal to 1. We wish to emphasize that the energy flow,  $w\vec{v}$ , given by (6), does not vanish, even though there is no heat conduction.

In the presence of viscosity, terms of second order in  $u$  appear in the energy equation (6), while a term linear in  $u$  appears in (7), the momentum equation. This means that the viscosity terms are relatively unimportant in the energy transport when  $u$  is small. The momentum equation, however, indicates that viscosity influences the time evolution of  $u$ . Thus viscosity can serve to disrupt the energy flow and generate entropy but cannot be a driving mechanism for energy removal.

The authors of Ref. [1] also list equation (6) with the high order viscosity terms dropped. This explicitly shows that energy can change in time in the absence of viscosity. In Ref. [1], however, a different equation with the dominant term  $\propto wu^i$  missing, and with the second order viscosity term present, is also given and used to describe the rate of energy change,

$$\Delta w \frac{dR}{dt} = \left(\frac{4}{3}\eta + \zeta\right)u_R \frac{du_R}{dr}. \quad (8)$$

The terms  $\eta$  and  $\zeta$  are the shear and bulk viscosity coefficients respectively, and the left hand side represents the energy flux density (energy per unit area per time).

Equations (8), and (6) are inconsistent. Equation (8) suggests that the energy flow is provided solely by viscous effects. It further suggests that, in the absence of such effects, the hadron bubble could not grow because the flow of the latent heat generated by the phase transition is forbidden. The fact that a  $\vec{v}$  is present, however, means there is energy flow  $\propto wv^i$ , regardless of viscous effects, and the fact that heat conduction can not be defined. Physically one may suggest that the energy is being carried away by quarks and antiquarks i.e. by convection to list a possible mechanism.

## III. SOLUTION OF RELATIVISTIC FLUID EQUATIONS FOR NONVISCOSITY CASE

We now consider solving explicitly for the rate  $\kappa$  in the absence of both viscosity and baryon number. This exercise, explicitly shows that  $\kappa$  does not vanish. In Section 5, we further suggest that the addition of viscous effects leads to small corrections in  $\kappa$ .

In the nonviscous case and in the limit of small velocities the relativistic energy and momentum differential equations become [5] (6) and (7). These equations have the same form whether the baryon number is zero or not.

The method of solution we use here was suggested by C. Goebel. The differential equations are solved in the quark vapor region only and are then related to the interior hadron material through boundary conditions. One

begins with the linearized relativistic equations in the quark region for a velocity field with spherical symmetry:

$$\begin{aligned}\partial_t \tilde{e} + \frac{1}{r^2} w_v \partial_r (r^2 v) &= 0 \\ w_v \partial_t v + \partial_r \tilde{p} &= 0.\end{aligned}\quad (9)$$

The boundary conditions or Kotchine conditions, across the interface are found by equating  $T^{0i}$  in the two regions in a frame in which the surface of discontinuity is at rest and also  $T^{ii}$  in the two regions with a surface tension term added. The results are:

$$\begin{aligned}p_h - p_v &= \frac{2\sigma}{R} \\ w_v(v_v - \dot{R}) &= w_h(v_h - \dot{R}).\end{aligned}\quad (10)$$

Use has been made of the Galilean transformation as  $v \ll 1$  and only terms linear in  $v$  retained. The  $v$ 's above represent velocities measured relative to the bubble center. Here the subscript  $v$  refers to the quark vapor and  $h$  to the hadron vapor.  $\dot{R}$  is the velocity of the bubble surface. For simplicity the velocity inside the hadron bubble,  $v_v$  is taken to be zero for the following solution. This condition is not an essential one (see Appendix A). A third boundary condition is needed to give information about the combustion rate. We postulate it as:

$$w_v(v_v - \dot{R}) = C\Delta(\delta T_h - \delta T_v).\quad (11)$$

Here  $C$  is a constant related to the efficiency of energy flow across the quark-hadron interface. For illustration purposes we take:

$$\delta T_h = \delta T_v\quad (12)$$

which is equivalent to  $C \rightarrow \infty$ . In a future work we examine the effect of finite  $C$ . Since both  $e$  and  $p$  are functions of the temperature,  $T$ , alone, the variations in  $e$  and  $p$ , i.e.,  $\tilde{e}$  and  $\tilde{p}$ , satisfy

$$\tilde{p} = c^2 \tilde{e}\quad (13)$$

where

$$c^2 = p'_v/e'_v,\quad (14)$$

the prime denoting the  $T$  derivative. It can be shown that  $c$  is the velocity of sound in the vapor. Next one can consider a variation of pressure away from the stationary solution,  $\tilde{p}$ . Then:

$$\tilde{p}_v - \tilde{p}_h = \frac{2\sigma \tilde{R}}{R_o^2}.\quad (15)$$

Here  $R_o$  is the critical size and  $\tilde{R}$  the variation from this size. The quantities are all evaluated at  $R_o$  in (10).

Given the following forms for vapor and hadron pressures:

$$\begin{aligned}p_v &= a_v T^4 - B \\ a_v &= \frac{37\pi^2}{90} \\ B &= 235^4 MeV^4 \\ p_h &= a_h T^4 \\ a_h &= \frac{3\pi^2}{90}.\end{aligned}\quad (16)$$

one finds:

$$\tilde{p}_v = -\frac{a_v}{a_v - a_h} \frac{2\sigma}{R_o^2} \tilde{R}\quad (17)$$

if one takes  $\delta T_v = \delta T_h$ . This particular choice provides for changes in pressure in both the hadron bubble and quark vapor regions as the bubble grows. Differentiating (13) with respect to time and using (17) one obtains a relation which holds at  $R_o$ :

$$\dot{\tilde{e}} = yz\quad (18)$$

where

$$y = \frac{2\sigma}{c^2 R_o^2} \frac{w_v}{(\Delta w)^2}\quad (19)$$

and

$$z = w_v v\quad (20)$$

if  $v_h$  is taken to zero as a first approximation (see Appendix A). In the vapor region (18) becomes a wave equation upon differentiation with respect to time and substitution of the second relation of (9) differentiated with respect to  $r$ :

$$\partial_t^2 \tilde{e} = c^2 \nabla^2 \tilde{e}.\quad (21)$$

Explicitly using:

$$\tilde{e} = \frac{f'(r - ct)}{r}\quad (22)$$

leads to:

$$z = \frac{c}{r^2} (rf' - f).\quad (23)$$

At this stage  $f$  is an arbitrary function and  $f'$  is its derivative with respect to  $r - ct$ . Using (18) one obtains:

$$f'' + yf' - \frac{y}{R_o} f = 0.\quad (24)$$

Solving with the form  $f(x) = \exp(\mu x)$  one obtains:

$$\mu = \frac{-1}{2} (y \pm \sqrt{(y^2 + \frac{4y}{R_o}}).\quad (25)$$

The upper sign is associated with the growing mode. Finally, the rate  $\kappa$  can be associated with  $-c\mu$ , so that

$$\kappa = c \frac{1}{2} (y + \sqrt{y^2 + \frac{4y}{R_o}}). \quad (26)$$

In the limit of small  $y$  one obtains:

$$\kappa = \sqrt{\frac{2\sigma w_v}{R_o^3 (\Delta w)^2}}. \quad (27)$$

For the purpose of comparison, we next review the solution provided in Ref. [3] for a nonrelativistic system with nonzero chemical potential. That work provides a  $\kappa$  for a drop (e.g. water) growing in a nonviscous vapor with perfect heat conduction. ( Note: There is an error in the second part of [3] which involves thermal conduction. The error is corrected in [7] and is unrelated to the first evaluation of  $\kappa$  in [3] which ignores changes in temperature and hence also the specific thermal conduction terms.) The relativistic equations (6,7) solved above are of the same form as the nonrelativistic mass conservation and momentum equations used in the first part of [3]. Thus one can compare the methods of solution and the results for the nonrelativistic treatment of [3] with the relativistic case.

The nonrelativistic equations of [3], expanded about a stationary solution, are:

$$\begin{aligned} \partial_t \nu &= -\nabla \cdot (\bar{n} \vec{u}) \\ \bar{n} \partial_t \vec{u} &= -\nabla \tilde{p}. \end{aligned} \quad (28)$$

Here  $\bar{n}$  is the stationary density and  $\nu$  and  $\vec{u}$  the changes in density and velocity from the stationary results. It is further assumed that the time dependence of all quantities is  $\exp(\kappa t)$ . The term  $\nabla \tilde{p}$  is obtained in [3] from the following expression for the free energy:

$$\begin{aligned} F &= F_K + F_I \\ F_K &= \frac{1}{2} \int d^3r m v^2 \\ F_I &= \int d^3r \left[ \frac{1}{2} K (\nabla n)^2 + f(n) \right] \end{aligned} \quad (29)$$

together with:

$$\frac{1}{n} p = \nabla \frac{\delta F_I}{\delta n} \quad (30)$$

where  $n$  is evaluated at  $\bar{n} + \nu$ . Equation (30) is consistent with the thermodynamical relation:

$$p = \mu N - f \quad (31)$$

where

$$\mu = \frac{\partial F}{\partial N} \quad (32)$$

holding  $T$  and  $V$  constant. Using

$$F = f\left(\frac{N}{V}\right)V \quad (33)$$

where  $N$  is constant one can obtain  $p$  from

$$p = -\frac{\partial F}{\partial V} \quad (34)$$

holding  $N$  and  $T$  constant. Thus

$$p = n \frac{\partial f}{\partial n} - f. \quad (35)$$

One may then obtain:

$$\partial_t \vec{u} = -\frac{1}{m} \nabla (-K \nabla^2 + \frac{\partial^2 f}{\partial \bar{n}^2}) \nu. \quad (36)$$

It is interesting to note that although the expressions for the pressure in terms of  $f$  are different for the zero-baryon number relativistic and nonzero baryon number nonrelativistic cases the change in pressure similar. For the two cases one has:

$$\begin{aligned} \tilde{p} &= c_1^2 \tilde{e} \\ \tilde{p} &= c_2^2 \tilde{n} \end{aligned} \quad (37)$$

respectively. In both cases  $c_i^2$  denotes the speed of sound in the vapor. For the nonrelativisitic case:

$$\begin{aligned} c_2^2 &= \frac{\partial p}{\partial n} \\ &= n \frac{\partial^2 f}{\partial n^2}. \end{aligned} \quad (38)$$

The first equation of (28) is of the same form as the first equation of (9) but the two have different origins. The latter is the result of energy conservation which follows from (4) involving the energy-stress tensor. The former is a result of particle conservation. There exists in relativistic fluid mechanics for systems with nonzero chemical potentials an equation similar to the first of (28) which describes the conservation of a net number of either particles or antiparticles. Such an equation only holds for nonzero baryon number and is not used in this paper which deals with the zero baryon number case only. As a result it is better not to look for a one-to-one correspondence between the variables of the relativisitic and nonrelativistic equations even though (28) and (9) are of similar form with  $n$  replacing  $w$ .

One can proceed to solve the wave equation resulting from a manipulation of (28) in all space instead of solving just in the exterior region and using the Kotchine boundary conditions as done in the method of section 3. The wave equation is straightforward to solve in the exterior and interior regions of the bubble. In the interface region, however, the solution is more difficult as  $\partial^2 f / \partial \bar{n}^2$  varies and Langer and Turski [3] solve it approximately using the assumption of small  $\kappa$ . Finally one may integrate  $\nu$  over all space and set this value equal to zero because the total amount of material is conserved. The integral in the interior contributes little and is ignored in [3]. The result of [3] then gives:

$$\kappa = \sqrt{\frac{2\sigma n_v}{mR_o^3(\Delta n)^2}}. \quad (39)$$

Here  $n_v$  is the vapor density. It is interesting to note that  $\kappa$  vanishes with vanishing  $n_v$ . The factor  $n_v/\Delta n$  enters the problem in the following way. In the interface region the solution of  $r\nu$  taken from [3] is equal to:

$$(\frac{\partial^2 f}{\partial \bar{n}^2})^{-1} = \frac{R^2 \Delta n}{2\sigma} \frac{d\bar{n}}{dr} \quad (40)$$

at  $r$  near  $R$ . Thus:

$$n_v \frac{\partial^2 f}{\partial \bar{n}^2} \nu = n_v \frac{2\sigma}{R^2 \Delta n} (\frac{d\bar{n}}{dr})^{-1} \nu. \quad (41)$$

The LHS is  $\delta p_v$  which is equivalent to  $\tilde{p}_v$  and the factor  $n_v/\Delta n$  is related to the expression for the pressure of the vapor in equilibrium with a drop in this case namely:

$$\begin{aligned} p_v(\text{total}) &= p_o + \delta p_v \\ &= p_o + \frac{2\sigma n_v}{R_o \Delta n} \end{aligned} \quad (42)$$

where  $p_o$  is the pressure of a system with a large amount of liquid in equilibrium with vapor. The term  $\delta p_v$  describes the change from this pressure due to the effects of the surface of the drop. The procedure of solving in the interface region simply ensures that the outside and inside pressures are related by:

$$\frac{2\sigma}{R}. \quad (43)$$

One may now attempt to apply the above ideas to the relativistic equations (6,7). These lead to:

$$\partial_t^2 \tilde{e} = -\nabla^2 \tilde{p}. \quad (44)$$

In the nonrelativistic and zero-baryon number relativistic problem  $\nabla \tilde{p}$  is equal to a constant multiplied by  $\nu$  and  $\tilde{e}$  respectively. Furthermore the same Kotchine equations (10) with  $w$ 's replaced with  $n$ 's apply to the nonrelativistic problem. As a consequence the low  $\kappa$  relativistic zero-baryon result must be the same as the nonrelativistic one with  $n$ 's replaced with  $w$ 's or namely:

$$\kappa = \sqrt{\frac{2\sigma w_v}{R^3(\Delta w)^2}}. \quad (45)$$

This is identical to (27).

#### IV. FREE ENERGY AND PRESSURE GRADIENTS

The work of Ref. [1] suggests that energy is removed during bubble growth by viscous effects alone (8). This result is already suggested by the authors' treatment in [1] of the relativistic momentum equation, i.e. the second

of (9) with an additional term linear in viscosity coefficients to describe viscous effects. The result of [1] that  $\kappa$  is proportional to the viscosity coefficients follows from the replacement of  $\nabla p$  in (9) with a particular expression involving the free energy of the bubble plus vapor system. We examine the relationship between pressure and free energy more closely to see how it leads to the conclusions of [1] concerning the role of viscosity.

The free energy is taken to have the form [8]:

$$\begin{aligned} F &= F_K + F_I \\ F_K &= \frac{1}{2} \int d^3 r w v^2 \\ F_I &= \int d^3 r [\frac{1}{2} K(\nabla e)^2 + f(e)]. \end{aligned} \quad (46)$$

Here  $K$  is a constant related to  $\sigma$ , the surface tension. The free energy is of the same form as that used in non-relativistic work [3] but is taken as a function of  $e$ , the energy density, as opposed to  $n$ , the number density. In the baryon-free case there is no number density. In [1] (equations 17-25) the pressure gradient term is associated with:

$$\begin{aligned} -' \nabla p' &= \frac{\delta F_K}{\delta e(\vec{r})} \nabla e \\ &= -K(\nabla^2 e) \nabla e + \frac{\partial f}{\partial e} \nabla e. \end{aligned} \quad (47)$$

It is important to note that the ' $\nabla p'$ ' is not simply a pressure but a combination of a pressure and a force term. Both of these terms are found in the Euler or Navier-Stokes equations. Such an identification must be made for the following reason. The pressure on the inside differs from that on the outside so there is necessarily a pressure gradient in the interface. If there were no force term then the Euler equation would require a changing fluid velocity in the equilibrium situation which is unphysical. The force term is associated with the  $K(\nabla e)^2$  term which is related to the surface tension by:

$$4\pi R^2 \sigma = K \int d^3 r (\nabla e)^2. \quad (48)$$

This is the term needed to balance the differing pressures. Identical considerations hold in the nonrelativistic case, with  $n$  taking the place of  $e$ .

In a region of  $e(\vec{r})$  varying slowly equation(47) suggests the identification:

$$\frac{\partial f}{\partial e} \nabla e = \nabla f = -\nabla p \quad (49)$$

and

$$p = -f. \quad (50)$$

The term proportional to  $K$  has been dropped in this region. We also assume this form.

Let us examine the approach used in [1]. There one linearizes the relativistic momentum equation for small variations,  $\tilde{e}$  for the energy and  $\vec{v}$  for the velocity, about the stationary solutions of  $\bar{e}$  and  $\vec{0}$  respectively for the hadron bubble in equilibrium with the quark vapor. Specifically:

$$-\nabla \tilde{p} = \frac{\partial f}{\partial \bar{e}} \nabla \tilde{e} + \frac{\partial^2 f}{\partial e^2} \tilde{e} \nabla \bar{e}. \quad (51)$$

The assumption:

$$\frac{\partial f}{\partial e} = 0 \quad (52)$$

is used. This is a consequence of the minimisation of (46) with respect to  $e$  to obtain a profile for the stationary solution  $\bar{e}$ . The stationary solution of [1] is described by the solution of:

$$\frac{\partial F_I}{\partial e} = -K \nabla^2 \bar{e} + \frac{\partial f}{\partial \bar{e}} = 0 \quad (53)$$

where a fourth order polynomial in  $e$  is chosen for  $f$ . This approach follows nonrelativistic work where  $n$  is used in place of  $e$ . In the nonrelativistic case, however, one constrains the number of particles present and so does not use:

$$\frac{\partial f}{\partial n} = 0. \quad (54)$$

With the use of (52) equation (64) of [1] :

$$\partial_t(\bar{w}\vec{v}) = (\nabla \bar{e})[-K(\nabla^2) + f'']\nu + \nabla[(\zeta + \frac{4}{3}\eta)\nabla \cdot \vec{v}] \quad (55)$$

is obtained. One next makes the association  $\partial_t \vec{v} = \kappa \vec{v}$ . Evaluating (55) in the quark vapor region where  $\bar{e}$  is constant causes the first term in (55) to vanish and suggests  $\kappa \propto$  viscosity coefficients.

Following this approach the velocity would not change in time and the bubble would not grow in the absence of viscosity. We evaluate (51) differently. The term:

$$\frac{\partial f}{\partial \bar{e}} \quad (56)$$

is not taken to be zero but rather  $c^2$ . This does not cause  $\nabla \tilde{p}$  to vanish and so bubble growth occurs with or without viscosity.

## V. EQUATIONS WITH VISCOSITY PRESENT

In the presence of viscosity a term  $\tau^{ik}$  is added to the tensor  $T^{ik}$  (3) where:

$$\begin{aligned} \tau^{ik} &= -\eta \left( \frac{\partial u^i}{\partial x^k} + \frac{\partial u^k}{\partial x^i} + u^k u^l \frac{\partial u^i}{\partial x^l} + u^i u^l \frac{\partial u^k}{\partial x^l} \right) \\ &\quad - (\zeta - \frac{2}{3}\eta) \frac{\partial u^l}{\partial x^l} (g^{ik} + u^i u^k). \end{aligned} \quad (57)$$

The energy and momentum fluid equations in the presence of viscosity, keeping only first order terms in  $v$  the velocity, are:

$$\begin{aligned} \partial_t e &= -\nabla \cdot w\vec{v} + O(v^2) \\ w\partial_t v &= -\nabla p + \nabla \left[ \left( \frac{4}{3}\eta + \zeta \right) \nabla \cdot \vec{v} \right] \end{aligned} \quad (58)$$

These are the same as those of [1]. For small viscosity coefficients and small  $\kappa$  it appears that the viscosity term is small compared to other terms as it involves two gradients. Here we assume a non-vanishing pressure gradient. From the equations with viscosity absent one can see that gradients of  $v$  are smaller than  $e$  by a factor of  $\kappa$ , and  $\kappa$  is assumed to be small. Next we consider the effect of viscosity on the boundary conditions. In the absence of viscosity the boundary conditions (10) can be expressed, in a frame moving with the interface, by

$$\begin{aligned} \frac{x_i x_j}{r^2} T_v^{ij} &= \frac{x_i x_j}{r^2} T_h^{ij} \\ \frac{x_i}{r} T_v^{0i} &= \frac{x_i}{r} T_h^{0i}. \end{aligned} \quad (59)$$

In the presence of viscosity:

$$T^{ij} = pg^{ij} + \tau^{ij} \quad (60)$$

to first order in velocity where only first order velocity terms in  $\tau$  are retained.

In a spherical problem one has

$$\begin{aligned} T^{ij} \frac{x_j x_i}{r^2} &= \\ &[p + \left( \frac{4}{3}\eta + \zeta \right) \nabla \cdot (v - \dot{R}) \\ &+ \frac{4\eta(v - \dot{R})}{r}] \end{aligned} \quad (61)$$

Thus  $p$  in the first of (10) is replaced with the RHS of (61) while the second boundary condition of (10) remains unchanged to first order in velocity. Here  $v$  is the velocity measured in a frame in which the bubble center is at rest. A Galilean transformation has been performed to relate this frame with that moving with the interface because all velocities are small compared to that of light.

We first consider the effect of the viscosity-modified transport equations. One can solve (58) explicitly following the method of section 3, using (13h) and replacing  $-\nabla \cdot \vec{v}$  in (58) with  $\partial_t e/w$ . The equation in the vapor becomes:

$$\partial_t^2 \tilde{e} = c^2 \nabla^2 \tilde{e} + \nabla^2 \left( \frac{\frac{4}{3}\eta + \zeta}{w_v} \right) \partial_t \tilde{e}. \quad (62)$$

For  $\frac{(\frac{4}{3}\eta + \zeta)}{w_v}$  constant in space one obtains:

$$\tilde{e} = \frac{A}{r} \exp(-qr + \kappa t) \quad (63)$$

where  $A$  is a constant and  $\kappa$  and  $q$  are related by:

$$\kappa^2 = c^2 q^2 + \frac{(\frac{4}{3}\eta + \zeta)}{w} \kappa q^2. \quad (64)$$

Thus in the vapor region one obtains the same result as (21) but with  $c^2$ , the speed of sound, replaced with:

$$c^2(1 + \frac{(\frac{4}{3}\eta + \zeta)}{w} \frac{\kappa}{c^2}). \quad (65)$$

Due to the presence of a  $\kappa$  factor the second term on the RHS of (64) is small compared to the first. This suggests that viscosity is a small effect. Solving the first of (9) which is the same for both the nonviscous and viscous situations an expression for  $v$ :

$$v = \frac{\kappa}{wq^2} A \left( \frac{q}{r} + \frac{1}{r^2} \right) \exp(-qr + \kappa t) \quad (66)$$

is found.

To see the explicit effect of viscosity on  $\kappa$  a relationship incorporating the boundary conditions is needed. Eq. (64) simply provides a relationship between two unknowns  $q$  and  $\kappa$ ; an additional equation is needed relating the two. To obtain such an expression (9) and (61) are combined in the same manner as (9) and (10) in section 3 to obtain an expression analogous to (17). The addition of viscosity modifies (17). To see how this modification arises we consider the derivation of (17) in section 3. To obtain the result (17) one solves for  $\delta T_h = \delta T_v \equiv \delta T$  in the first equation of (10) by setting:

$$\tilde{p}_v \propto a_v \delta T_v \quad (67)$$

and taking a similar expression for  $\tilde{p}_h$ . One then substitutes the value for  $\delta T$  in (67) to obtain  $\tilde{p}_v$ . In the presence of viscosity a similar method is employed but now  $u_h, u_v, \dot{R}$  in (61) must be replaced with expressions involving  $\delta T$ . This allows one to solve for  $\delta T$  and thus  $\tilde{p}_v$ . Terms with  $\nabla \cdot v$  can be replaced with terms involving  $\kappa \tilde{e}_v$  using the first of (58). Terms involving  $\nabla \cdot \dot{R}$  are proportional to  $\kappa \tilde{e}_v$  because the second of (10) relates  $\dot{R}$  to  $v_v$ . Terms involving  $v_h$  are dropped to be consistent with section 3 (see Appendix A). Finally one must consider terms with  $\dot{R}$  alone. Given (63) and (66) it is possible to relate  $v_v$  and  $\tilde{e}_v$  at the interface to obtain:

$$v_v = \frac{\kappa}{q^2 w_v} \left( q + \frac{1}{R} \right) \tilde{e}_v. \quad (68)$$

The use of (61) essentially leads to the replacement of the factor:

$$\frac{a_v}{a_v - a_h} = \frac{w_v}{\Delta w} \quad (69)$$

appearing in (17) with the factor

$$x \equiv w_v / [w_h - w_v \left( 1 + \frac{3f_v \kappa}{\Delta w} - \frac{3f_q \kappa}{\Delta w} + \frac{3f_q \kappa}{w_q} + \frac{12c^2 N^2}{R \Delta w w_v \kappa} (\eta_h w_v - \eta_v w_h) \left( \frac{\kappa}{cN} + \frac{1}{R_o} \right) \right)]. \quad (70)$$

Here

$$\begin{aligned} f_h &= \frac{4}{3} \eta_h + \zeta_h \\ f_v &= \frac{4}{3} \eta_v + \zeta_v \\ N &= \sqrt{\left( 1 + \frac{(\frac{4}{3}\eta + \zeta)}{wc^2} \kappa \right)}. \end{aligned} \quad (71)$$

Next Eq. (13) is differentiated with respect to time to obtain an expression for  $\dot{p}_v$ . The term  $\dot{p}$  (13) is replaced with an expression involving  $\dot{R}$  which is in turn replaced with an expression in the velocity  $v$ . The term  $\dot{e}$  is written in terms of the gradient of the velocity. One then obtains:

$$-\frac{1}{r^2} w_v \partial_r r^2 v = Y w_v v \quad (72)$$

where:

$$Y = yx \frac{\Delta w}{w_v}, \quad (73)$$

$y$  being given in (19). The velocity in the hadron material is again neglected.

Substituting into (72) yields:

$$Y \left( q + \frac{1}{R_o} \right) = q^2 \quad (74)$$

which may be compared with (26), the result in the absence of viscosity. Eq. (74) must be solved with  $q$  expressed in terms of  $\kappa$  namely:

$$q = \frac{\kappa}{cN}. \quad (75)$$

The modification due to viscosity resulting from the change in the transport equations (58) is present in  $q$  while that due to the change in the boundary conditions (61) is contained in the replacement of  $y$  with  $Y$ . In general (74) must be solved numerically for  $\kappa$  as it is of higher than third order in  $\kappa$ . An approximate solution is obtained if the  $q$  term can be neglected compared to  $\frac{1}{R_o}$  in (74):

$$\kappa = cN \sqrt{\frac{Y}{R_o}} \quad (76)$$

with  $Y$  and  $N$  being evaluated with  $\kappa$  replaced with  $\kappa_o$ , the result in the absence of viscosity (25). The solution for  $\kappa$  in the absence of viscosity is of the same form as (76) except  $N$  is replaced with 1 and  $Y$  with  $y$ .

## VI. CONCLUSION

We have examined the growth of a hadron bubble in quark vapor and suggest that the physical process is different from that presented in [1]. Specifically we suggest

that the formalism of relativistic hydrodynamics permits energy flow in the absence of heat conduction and viscosity. We have solved the relativistic fluid equations for a system with zero baryon number (no thermal conduction) and no viscosity and have obtained a nonzero value for  $\kappa$ . We have compared the relativistic fluid equations and resulting  $\kappa$  with the nonrelativistic equations and  $\kappa$  presented in [3]. The method used in [3] is approximate in that it only holds in the limit of small  $\kappa$  and differs from that presented in this paper which provides an exact result for  $\kappa$ . Finally we have solved the relativistic fluid equations for zero baryon number in the presence of viscosity. We suggest that viscous effects introduce only a small correction in the nonviscous value of  $\kappa$  and provide an expression for the correction.

In this paper we have used a specific set of boundary conditions linking the quark and hadron regions. Investigation of different sets of boundary conditions and their effect on  $\kappa$  is in progress. In addition we consider the calculation of  $\kappa$  for a medium with nonzero baryon number and thus heat conduction.

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## APPENDIX A: A

A solution for  $\kappa$  has been obtained under the assumption that  $\delta T_h = \delta T_q$  at the surface of the bubble and that the velocity in the hadron material is zero. One may solve for  $\kappa$  without the latter restriction to find a more general result which in the limit of small  $\kappa$  reduces to the zero velocity solution. For small  $\kappa$  the velocity on the inside is much less than that on the outside and so can be ignored. One may note that the authors of [3], who consider the problem of nonrelativistic drop forming in a vapor, also disregard the velocity inside the drop in the limit of small  $\kappa$ .

To solve the general problem with velocity retained in both the bubble and vapor regions the hydrodynamical equations must be solved in both regions and use made of the full boundary condition:

$$w_q v_q - w_q \dot{R} = w_h v_h - w_h \dot{R}. \quad (\text{A1})$$

The solutions in the inner region are:

$$\begin{aligned} \tilde{e}_h &= C_h \frac{\sinh(qr)}{\sinh(qR_o)} \frac{e^{\kappa t}}{r} \\ v_h &= \frac{C_h}{\sinh(qR_o)} \frac{\kappa}{w_h q^2} \left( -q \cosh(qr) + \frac{\sinh(qr)}{r} \right) \frac{e^{\kappa t}}{r}. \end{aligned} \quad (\text{A2})$$

In addition there is the relation:

$$\kappa = qc_h \quad (\text{A3})$$

where  $c_h$  is the speed of sound in the hadron material and equals that of the quark medium. The solution in the quark region is of the form:

$$\tilde{e}_q = C_q \frac{\exp[\kappa t - q(r - R_o)]}{r}. \quad (\text{A4})$$

Thus one may obtain the relationship:

$$\begin{aligned} \frac{\tilde{e}_q}{\tilde{e}_h} &= \frac{a_q \delta T_q}{a_h \delta T_h} \\ &= \frac{C_q}{C_h} \end{aligned} \quad (\text{A5})$$

at the interface surface. Finally one may solve for  $\kappa$  by evaluation of:

$$\begin{aligned} \dot{\tilde{p}}_q &= c^2 \kappa \tilde{e}_q \\ &= \frac{2\sigma}{R^2 \Delta w} (w_q v_q - w_h v_h) \end{aligned} \quad (\text{A6})$$

with (A3). All quantities are evaluated at the surface of the bubble. The resulting expression for  $\kappa$  is complicated due to the presence of the terms  $\cosh qR$  and  $\sinh qR$ . One may, however, consider the limit of small  $q$  which is equivalent to small  $\kappa$ . In that case:

$$v_h \propto -\kappa R. \quad (\text{A7})$$

Here the  $\cosh$  and  $\sinh$  terms have been expanded to second and third powers of  $q$  respectively. It is possible to see that  $v_h$  is much smaller than the expression for  $v_q$  in the case of small  $q$  and so  $v_h = 0$  as an approximation. One can proceed to solve for  $\kappa$  with the next order approximation using (A7) instead of  $v_h = 0$ . Then one must solve the quadratic:

$$c^2 q^2 = \frac{2\sigma w_q}{R^2 (\Delta w)^2} \left( q + \frac{1}{R} + \frac{a_h R q^2}{a_q^3} \right) \quad (\text{A8})$$

to obtain:

$$q = \frac{1}{2c^2 \left( -y \frac{a_h R}{a_q^3} \right)} \left[ y c^2 + \sqrt{c^4 y^2 - \frac{4y c^4}{R} \left( 1 - y \frac{a_h R}{3a_q} \right)} \right]. \quad (\text{A9})$$

Here  $y$  is given by (19). In the limit  $y \ll \frac{4}{R}$  one obtains the results of section 3.

## APPENDIX B: B

A method to solve for  $\kappa$  using differential equations in a metastable and stable phase region together with Kotchine conditions relating the two phase at the bubble surface is presented in section 3. This method holds  $\sigma$ , the surface tension, fixed as the bubble grows. An alternative method, which yields identical results, is considered in this appendix. It is based on a modification of

Langer and Turski's approach to the nonrelativistic problem. In the Langer and Turski approach, as outlined in section 3, one solves a wave equation for  $\tilde{n}$  in the vapor and interior drop regions and obtains solutions for  $\tilde{n}$  and  $v$  in these two regions. These solutions are functions of  $\kappa$ . Next an approximate solution for  $\tilde{n}$  is obtained in the interface region assuming small values of  $\kappa$ . Finally the relation:

$$\int d\vec{r}\tilde{n}(r) = 0, \quad (\text{B1})$$

namely particle conservation is used to obtain a value for  $\kappa$ . In [3] only low order terms in  $\kappa$  are retained after integration. If one, however, evaluates the expression exactly and considers the manipulations in the interface to also be exact one obtains the same results as in section 3. Specifically, in the nonrelativistic problem one has:

$$\partial_t \tilde{n} = -\nabla \cdot nv. \quad (\text{B2})$$

The condition (B1) is equivalent to:

$$\int d\vec{r}\partial_t \tilde{n} = 0 \quad (\text{B3})$$

for the case of interest namely:

$$\partial_t \tilde{n} = \kappa \tilde{n}. \quad (\text{B4})$$

The volume of all space can now be separated into three regions. Two of the regions are the inner drop and outer vapor regions and the third is the vicinity of the surface in which the density is changing. Using (B3) and Gauss' law one may make the identification:

$$\int \partial_t \tilde{n} = \text{interfacepiece} + n_{in}v_{in} - n_{out}v_{out}. \quad (\text{B5})$$

The surface piece is equivalent to:

$$(\Delta n)\tilde{R}. \quad (\text{B6})$$

This describes the change in the interface as equivalent to a drop with a sharp surface growing by the amount  $\tilde{R}$ . The inner and outer areas defining the interface are taken as equal. An expression for  $\tilde{R}$  can be obtained from:

$$\tilde{p}_{vapor} = -\frac{2\sigma n_v}{R^2(\Delta n)^2} \tilde{R} \quad (\text{B7})$$

and

$$\tilde{p}_v = c^2 \tilde{n}. \quad (\text{B8})$$

Here the Kotchine condition relating pressures on the inside and outside regions has been used. Langer and Turski in [3] obtain a similar result by approximately solving the transport equation:

$$m\kappa^2 \tilde{n} = \nabla \cdot n \nabla (-K \nabla^2 + \frac{\partial^2 f}{\partial n^2}) \tilde{n} \quad (\text{B9})$$

in the region in which the density is changing. The result of [3] keeps only terms to first order in  $\kappa$  but is equivalent to (B6) and (B7). Thus one finally obtains:

$$0 = n_{in}v_{in} - n_{out}v_{out} + \frac{c^2 \tilde{n} R^2 \Delta n}{2\sigma n_v}. \quad (\text{B10})$$

which is identical to the result of section 3 obtained using a method which utilised the Kotchine expressions and  $\tilde{n}$  and  $v$  solutions together with:

$$\tilde{p} = c^2 \tilde{n} \quad (\text{B11})$$

Identical arguments hold for the relativistic problem. For the relativistic problem to first order in the energy flow velocity over the speed of light one has:

$$\partial_t \tilde{e} = -\nabla \cdot w v. \quad (\text{B12})$$

This equation holds with or without viscosity for both baryon and baryon-free materials. Separating the volume again into three regions and using Gauss' law leads to the energy conservation condition:

$$w_h v_h - w_q v_q = \Delta w \dot{R}. \quad (\text{B13})$$

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